ON THE S_2-FICATION OF SOME TORIC VARIETIES

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First version in 2004, revised version march 2006

Abstract. In this paper we prove:

1. *Some results on the Cohen-Macaulayness of the canonical module.*
2. *We study the S_2-fication of rings which are quotients by lattices ideals.*
3. *Given a simplicial lattice ideal of codimension two I, its Macaulayfication is given explicitly from a system of generators of I.*

Introduction

Let X be an algebraic variety, the set of points where X is not Cohen-Macaulay is the Non-Cohen-Macaulay locus, this set was study in [1]. Macaulayfication is an analogous operation to resolution of singularities and was considered in [7] where the main theorem of Macaulayfication is given.

For any affine semigroup (without torsion) $S \subset \mathbb{N}^n$ let $G(S)$ be the subgroup of $\mathbb{Z}^n$ generated by S and $\bar{S}$ be the saturation of S inside $G(S)$, that is

$$\bar{S} = \{m \in G(S) : rm \in S \text{ for some } r \in \mathbb{N}\},$$

it is well known that the normalization of the semigroup ring $K[S]$ is given by $K[\bar{S}]$ and Hochster proved in [6] that $K[\bar{S}]$ is always a Cohen-Macaulay ring. We have an exact sequence:

$$0 \longrightarrow K[S] \longrightarrow K[\bar{S}] \longrightarrow K[\bar{S} \setminus S] \longrightarrow 0,$$

and $K[\bar{S}]$ is a Cohen-Macaulay ring containing $K[S]$, with the same ring of fractions. In general, the support of $K[\bar{S} \setminus S]$ does not coincide with the Non Cohen-Macaulay locus of $K[S]$ because $\bar{S}$ is too big. Our problem consist to look for a "minimal" subsemigroup $\tilde{S} \subset \bar{S}$ containing S such that $K[\tilde{S}]$ is a Cohen-Macaulay ring. In [5] and [10] the authors consider a semigroup $S' \subset \bar{S}$ which contains S such that we have an exact sequence:

$$0 \longrightarrow K[S] \longrightarrow K[S'] \longrightarrow K[S' \setminus S] \longrightarrow 0,$$

and $\dim K[S' \setminus S] \leq n-2$. $K[S']$ is the S_2-fication of $K[S]$. When $K[S']$ is a Cohen-Macaulay ring, the support of $K[\bar{S} \setminus S]$ coincide with the Non Cohen-Macaulay locus of $K[S]$. This is the case notably when S is a simplicial semigroup. The purpose of this paper is to give effective methods to compute the S_2-fication for a class of toric varieties. In the first part of this paper we consider the S_2-fication and give some general results on the Cohen Macaulayness of the canonical module, one

of them extends and improves Proposition 2.5 of [4]. We also extend and improve to the lattice case the above results from [5] and [10], given shorter proofs.

In the second part we consider a codimension two simplicial toric ring $K[S]$, and describe the Macaulayfication of this ring in terms of the system of generators of its ideal of definition as described in [8], this ideal can be computed by an effective algorithm which works in polynomial time at very low cost. This is also implemented in my software codim2simplicial, which computes the generators of a simplicial codimension 2 lattice ideal without using Groebner basis.

During the meeting Current trends in Commutative Algebra held in Levico, Italy, in June 2002, I have submitted to Peter Schenzel, the problem developed in this paper in sections two to four, then we have started a joint work on this subject during more than one year. Peter Schenzel got a proof using spectral sequences and decided to publish by himself in [13]. My proof developed here is completely different and elementary, it is a complement to Schenzel's proof.

1 Known results on local cohomology

The following results are well known [11], [12] section 1.2. All this results are also true for graded ring and modules.

Let (R, Q) be a Gorenstein local ring of dimension n, let (A, m) be a factor ring of R and M a finitely generated $A-$module of dimension d.

We recall the local duality's theorem:

Theorem 1 *We have an isomorphism :*

$$H^i_{\mathrm{m}}(M) \simeq H^i_Q(M) \simeq \mathrm{Hom}_R(\mathrm{Ext}^{n-i}_R(M, R), E(R/Q))$$

We denote by $D^i(M)$ the finitely generated R-module $\mathrm{Ext}^{n-i}_R(M, R)$, and we set by $K_M = D^d(M)$ the canonical module. We recall some of their properties:

1. For any exact sequence
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
we have a long exact sequence:
$$\ldots \longrightarrow D^i(M'') \longrightarrow D^i(M) \longrightarrow D^i(M') \longrightarrow D^{i-1}(M'') \longrightarrow D^{i-1}(M) \longrightarrow D^{i-1}(M') \longrightarrow$$

2. $D^i(M) = 0$ for either $i > d$ or $i < 0$, $D^d(M)$ has dimension d. Moreover depth $D^d(M) \geq \min\{d, 2\}$, $D^d(M)$ satisfies the condition S_2 when $d \geq 2$, and if M is Cohen-Macaulay then so is $D^d(M)$.

3. For all $P \in \mathrm{Supp}\ M$ we have $(D^d(M))_P = D^d(M_P)$.

4. $\dim D^i(M) \leq i$ for all $0 \leq i < d$. Suppose in addition that M is equidimensional. Then M satisfies the condition S_k if and only if $\dim D^i(M) \leq i - k$ for all $0 \leq i < d$.

5. If M is unmixed and $d \geq 2$, then we have an exact sequence :
$$0 \longrightarrow M \longrightarrow D^d(D^d(M)) \longrightarrow N \longrightarrow 0$$
where $\dim N \leq \dim M - 2$. Moreover M satisfies the condition S_2 if and only if M is isomorphic to $D^d(D^d(M))$.

2 One result on the canonical module

Theorem 2 *Let (A, m) be a factor ring of a Gorenstein local ring, let M be a finitely generated $A-$module of dimension d.*

1. *Assume that $d \geq 3$ and* depth $(M) > 0$, *then* depth $D^{d-1}(M) = 0$ *if and only if* depth $K_M = 2$.
2. *Assume that $d \geq 2$,* depth $(M) = d-1$, *and $D^{d-1}(M)$ has dimension $d-2$. Then* depth $D^{d-1}(M) =$ depth $K_M - 2$.

In particular suppose that depth $(M) = d-1$, *and* $\dim D^{d-1}(M) = d-2$. *Then $D^{d-1}(M)$ is a Cohen-Macaulay module if and only if the canonical module K_M is Cohen-Macaulay.*

Let $a \in m$ be a non zero divisor of M. From the exact sequence :

$$0 \longrightarrow M \xrightarrow{\times a} M \longrightarrow M/aM \longrightarrow 0$$

we get the following long exact sequence:

$$0 \to D^d(M) \xrightarrow{\times a} D^d(M) \xrightarrow{\alpha} D^{d-1}(M/aM) \xrightarrow{\beta} D^{d-1}(M) \xrightarrow{\times a} D^{d-1}(M) \to D^{d-2}(M/aM) \to D^{d-2}(M) \to ...$$

From this exact sequence we get the short exact sequences:

$$0 \longrightarrow D^d(M) \xrightarrow{\times a} D^d(M) \longrightarrow \mathrm{Im}\ \alpha \longrightarrow 0 \qquad (1)$$

$$0 \longrightarrow \mathrm{Im}\ \alpha \longrightarrow D^{d-1}(M/aM) \longrightarrow \mathrm{Im}\ \beta \longrightarrow 0 \qquad (2)$$

Note that $\mathrm{Im}\ \beta = (0 :_{D^{d-1}(M)} a)$. From the exact sequence 2, we have the long local cohomology sequence:

$$0 \longrightarrow H^0_{\mathrm{m}}(\mathrm{Im}\ \alpha) \longrightarrow H^0_{\mathrm{m}}(D^{d-1}(M/aM)) \longrightarrow H^0_{\mathrm{m}}((0 :_{D^{d-1}(M)} a)) \longrightarrow H^1_{\mathrm{m}}(\mathrm{Im}\ \alpha) \longrightarrow 0,$$

where $H^0_{\mathrm{m}}(D^{d-1}(M/aM)) = H^1_{\mathrm{m}}(D^{d-1}(M/aM)) = 0$, since $\dim M/aM = d-1 \geq 2$ and $D^{d-1}(M/aM)$ satisfies condition S_2, hence the map $H^0_{\mathrm{m}}((0 :_{D^{d-1}(M)} a)) \longrightarrow H^1_{\mathrm{m}}(\mathrm{Im}\ \alpha)$ is an isomorphism.

1. If depth $K_M = 2$, suppose first that depth $D^{d-1}(M) > 0$, then we can choose $a \in m$, a non zero divisor for depth $D^{d-1}(M)$, this will imply that $H^1_{\mathrm{m}}(\mathrm{Im}\ \alpha) = 0$ and then depth $K_M \geq 3$. A contradiction.
   If depth $D^{d-1}(M) = 0$ we have either $\dim D^{d-1}(M) = 0$ or not. If $\dim D^{d-1}(M) = 0$ then the module $(0 :_{D^{d-1}(M)} a)$ is non null but has also dimension zero. If $\dim D^{d-1}(M) > 0$ then choose $a \notin \cup_{P \in \mathrm{Ass}\ (D^{d-1}(M)) \setminus \{\mathrm{m}\}} P$, we will have that $\dim(0 :_{D^{d-1}(M)} a) = 0$ and is non null. In both cases $H^1_{\mathrm{m}}(\mathrm{Im}\ \alpha) \simeq H^0_{\mathrm{m}}((0 :_{D^{d-1}(M)} a)) \neq 0$ and so depth $K_M = 2$.
2. We will prove the claim by induction on d. Remark that if $\dim M = 2$, our statement is true. In fact following Section 1, the canonical module is Cohen-Macaulay of dimension two and since by our hypothesis $D^1(M)$ is of dimension 0, it is Cohen Macaulay. Let $d \geq 3$, by the first claim we can assume that depth $D^{d-1}(M) > 0$.

   Let $a \in m$ be a non zero divisor for both M and $D^{d-1}(M)$. Since a is a non zero divisor for $D^{d-1}(M)$, we have $\beta = 0$ and we get two exact sequences:

$$0 \longrightarrow D^d(M) \xrightarrow{\times a} D^d(M) \xrightarrow{\alpha} D^{d-1}(M/aM) \longrightarrow 0$$

$$0 \longrightarrow D^{d-1}(M) \xrightarrow{\times a} D^{d-1}(M) \longrightarrow D^{d-2}(M/aM) \longrightarrow 0$$

It then follows that M/aM satisfies the induction hypothesis. Hence depth $D^{d-1}(M/aM) =$ depth $D^{d-2}(M/aM)+2$, the above two short exact sequences imply then that depth $D^d(M) =$ depth $D^{d-1}(M) + 2$.

This ends the proof of the theorem. As a consequence of the proof we have:

Corollary 1 *Let (A, m) be a factor ring of a Gorenstein local ring, let M be a finitely generated $A-$module with* $\dim M = d \geq 3$ *and* depth $(M) > 0$. *If* $\dim D^{d-1}(M) > 0$, *let $a \in m$ be a non zero divisor of M and $a \notin \cup_{P\in \mathrm{Ass}\ (D^{d-1}(M))\setminus\{\mathrm{m}\}} P$, then K_M/aK_M is isomorphic to $K_{M/aM}$ if and only if* depth $K_M \geq 3$. *In particular if K_M is a Cohen-Macaulay module then $K_{M/aM}$ is a Cohen-Macaulay module.*

Proof .- With the above notations, K_M/aK_M is isomorphic to $K_{M/aM}$ if and only if $\mathrm{Im}\ \beta = (0 :_{D^{d-1}(M)} a) = 0$. By our choice of a, we have that $\dim(0 :_{D^{d-1}(M)} a) = 0$, so $H^0_{\mathrm{m}}((0 :_{D^{d-1}(M)} a)) = (0 :_{D^{d-1}(M)} a)$ and then $\mathrm{Im}\ \beta = 0$ if and only if $H^1_{\mathrm{m}}(\mathrm{Im}\ \alpha) = 0$, this is equivalent to depth $K_M \geq 3$.

Example 1 *(See also[10]) Consider the semigroup in $S \subset \mathbb{N}^3$ generated by the elements* $(3, 0, 0), (2, 1, 0), (0, 3, 0), (3, 0, 1), (2, 1, 1), (0, 3, 1)$, *the semigroup ring $K[S]$ has dimension 3, codimension 3, and* depth $K[S] = 2$, *the ring $K[S]$ satisfies the condition S_2 of Serre so it is isomorphic to $D^3(D^3(K[S]))$. The canonical module $D^3(K[S])$ is not Cohen-Macaulay. Remark that we have* $\dim D^2(K[S]) = 0$.

3 S_2-fication of unmixed modules

Let (A, m) be a noetherian local ring, (resp. graded), quotient of a Gorenstein local ring (resp. graded Gorenstein ring) and M be an $A-$module of dimension d.

We recall that if M is unmixed, the module $D^d(D^d(M))$ satisfies the condition S_2 and we have an exact sequence :

$$0 \longrightarrow M \longrightarrow D^d(D^d(M)) \longrightarrow M'' \longrightarrow 0$$

with $\dim M'' \leq d - 2$. Moreover if there exist an $A-$module M' of dimension d, satisfying the condition S_2 and an exact sequence :

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'/M \longrightarrow 0$$

with $\dim M'/M \leq d - 2$, then $M' \simeq D^d(D^d(M))$. The $A-$module M' is the S_2-fication of M and if M' is a Cohen-Macaulay module it is a Macaulayfication of M.

Lemma 1 *Set $M' := D^d(D^d(M))$. Assume that M is unmixed not satisfying the condition S_2, then:*

A) *The canonical module $K_M = D^d(M)$ is a Cohen-Macaulay module if and only if M' it is.*

B) *If K_M is a Cohen-Macaulay module, then:*

$H^{i-1}_{\mathrm{m}}(M'/M) \simeq H^i_{\mathrm{m}}(M)$ *for $i = 1, \ldots, d-1$. In particular* depth $(M'/M) =$ depth $M - 1$ *and* $\dim M'/M = \max\{i \leq d - 2\ /\ H^{i+1}_{\mathrm{m}}(M) \neq 0\}$

As a special case M'/M is a Cohen-Macaulay Module if and only if only one of the local cohomology modules $H^i_{\mathrm{m}}(M), i \leq n - 1$, does not vanish.

In this case the Matlis dual $D^i(M)$ of $H^i_m(M)$ is a Cohen Macaulay module of dimension $i-1$.

In particular if depth $M = d-1$, *then M'/M is a Cohen-Macaulay Module of dimension $d-2$, and $D^{d-1}(M)$ is a Cohen-Macaulay module of dimension $d - 2$.*

C) *The Non-Cohen-Macaulay locus of M is given by* Supp (M'/M).

Proof .-

A) Since $\dim M'/M \leq n-2$ we have $D^d(M) \simeq D^d(M')$, if M' is Cohen-Macaulay, then $D^d(M')$ is a Cohen-Macaulay module, hence the canonical module K_M is Cohen-Macaulay of dimension d. The converse follows since $M' \simeq D^d(D^d(M'))$

B) From the long exact sequence of the local cohomology associated to the sequence:

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'/M \longrightarrow 0$$

we get $H^{i-1}_{\mathrm{m}}(M'/M) \simeq H^i_{\mathrm{m}}(M)$ for $i = 1, \ldots, d-1$, which implies B).

C) The above exact sequence is still exact by localization on any prime ideal P; on the other hand $K(K(M))_P = K(K(M_P))$ and recall that if M is Cohen-Macaulay then the natural map $M \longrightarrow K(K(M))$ is an isomorphism. It follows that the Non-Cohen-Macaulay locus of M is given by Supp (M'/M).

When M is unmixed we get a new version of theorem 2:

Theorem 3 *Let M be unmixed of dimension d, not satisfying the condition S_2, and* depth $M = d-1$, *then* $\dim D^{d-1}(M) = d-2$, *and* depth $D^{d-1}(M) =$ depth $K_M - 2$. *In particular $D^{d-1}(M)$ is Cohen-Macaulay if and only if K_M is Cohen-Macaulay.*

Proof .- In regard of Theorem 2, we need only to prove that $\dim D^{d-1}(M) = d-2$.

Set $M' = D^d(D^d(M))$, from the exact sequence :

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'/M \longrightarrow 0$$

with $\dim M'/M \leq d-2$, we have $D^d(M) \simeq D^d(M')$. Since M' satisfies S_2 we have $\dim D^{d-j}(M') \leq d-j-2$, for all $0 \leq j \leq d-1$.

Assume that $\dim M'/M = d-k < d-2$, for some $3 \leq k \leq d-1$, then from the long exact sequence associated to the above short exact sequence we have:

$$0 \longrightarrow D^{d-k}(M'/M) \longrightarrow D^{d-k}(M') \longrightarrow 0,$$

this carries a contradiction since $d-k = \dim M'/M = \dim D^{d-k}(M') = d-k \leq d-k-2$.

From the exact sequence

$$0 \longrightarrow D^{d-1}(M') \longrightarrow D^{d-1}(M) \longrightarrow D^{d-2}(M'/M) \longrightarrow D^{d-2}(M') \longrightarrow 0$$

we get $\dim D^{d-1}(M) = \dim D^{d-2}(M'/M) = d-2$ since $\dim D^{d-j}(M') \leq d-j-2$, for all $0 \leq j \leq d-1$.

Remark 1 *Let M be a finitely generated graded module over a ring of polynomials, with* $\dim M = d$, *and* depth $M = d-1$. *It is well known that if $0 \longrightarrow G \xrightarrow{\phi} F \longrightarrow \ldots$ is the last term of the minimal syzygies of M then $\ldots \longrightarrow F \xrightarrow{\sigma} G \longrightarrow D^{d-1}(M) \longrightarrow 0$ is a presentation of $D^{d-1}(M)$, where σ is the matrix transpose of ϕ.*

Example 2 *Let A be the affine ring of the projective surface in P^4 defined parametrically by:*

$$a = s^4 + t^4, b = s^2tu, c = s^3t, d = st^3, e = su^3$$

then $depth A = 2$ and $\sigma = (a, c^2 + d^2, ce, b^3)$. It follows that $D^2(A)$ is Cohen-Macaulay of dimension 1, and the S_2-fication is in fact a Macaulayfication. It is not difficult to check that

$$A' = K[s^4 + t^4, s^2tu, s^3t, st^3, su^3, s^2t^2]$$

is the Macaulayfication of A.

Example 3 *Let A be the affine ring of the projective surface in P^4 defined parametrically by:*

$$a = s^4, b = s^3t + u^4, c = s^2t^2, d = su^3, e = t^2u^2$$

a quick computation with Macaulay, if $char(K) \neq 2, 3$, gives that $depth A = 2$ and σ is given by

$$\begin{pmatrix} 0 & -ae & d^2 & -c & b & 0 & 0 \\ 2ab^3 - 2a^2bc + d^4 & 3/2ab^3 + 1/2a^2bc & a^2b^2 - a^3c & -3/2bd^2 - a^2e & -1/2ad^2 & e & 6c \end{pmatrix},$$

in this case m *is an associated prime ideal of the ideal generated by the entries of the second row of σ, but again using Macaulay we get that that $D^2(A)$ is Cohen-Macaulay of dimension 1. Also in this example the S_2-fication is in fact a Macaulayfication. We can check that*

$$A' = K[s^4, s^3t + u^4, s^2t^2, su^3, t^2u^2, s^2u^2]$$

is the Macaulayfication of A.

4 Lattice and toric ideals

Let $R = K[x_1, \ldots, x_m]$ be a polynomial ring, $L \subset \mathbb{Z}^m$ a lattice of rank r. We assume that L is a positive lattice, that is, every non zero vector in L has positive and negative coordinates. We can write every vector $\mathbf{u}$ in $\mathbb{Z}^m$ uniquely as $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$, where $\mathbf{u}_+$ and $\mathbf{u}_-$ are non-negative and have disjoint support. Set $I_L \subset R$ be the ideal generated by all the binomials $x^{\mathbf{u}_+} - x^{\mathbf{u}_-}$, where $\mathbf{u}$ runs over all vectors of L. $I_L \subset R$ is called a lattice ideal associated to L. Let Sat $L = \{\mathbf{u} \in \mathbb{Z}^m \mid k\mathbf{u} \in L$ for some $k \in \mathbb{Z}\}$. The group $\mathbb{Z}^m/L$ is a finitely generated abelian group, and $I_L \subset R$ is a prime ideal if and only if $\mathbb{Z}^m/L$ has no torsion. We quote the following theorem from the proof of Corollaries 2.2 and 2.5 of [3]:

Theorem 4 *Let K be an algebraically closed field of any characteristic $p \geq 0$. The ideal $I_L \subset R$ is always unmixed. Moreover any x_i is a non zero divisor modulo I_L.*

When the ideal $I_L \subset R$ is prime it is called toric. In the toric case the lattice L is usually viewed as the lattice of the relations of a finitely generated semigroup $S \subset \mathbb{N}^n$. In general we have an isomorphism $\mathbb{Z}^m/L \longrightarrow \mathbb{Z}^d \oplus H$, where H is a finite group, the images $\mathbf{a_1} \ldots, \mathbf{a_m}$ of the canonical basis of $\mathbb{Z}^m$ under this isomorphism generate a finitely generated semigroup $S \subset \mathbb{Z}^n \oplus H$, which generates $G(S) := \mathbb{Z}^d \oplus H$. In fact $K[S] := R/I_L = \oplus_{g \in S} K \underline{t}^{g_1} \underline{u}^{g_2}$, where $g_1 \in \mathbb{Z}^n, g_2 \in H$ and $g = (g_1, g_2)$.

We set $\tilde{S}$ the projection of S in $\mathbb{Z}^d$, let $\mathcal{C}_S$ be the cone generated by $\tilde{S}$ in $\mathbb{Q}^d$, and $F_1, F_2, \ldots, F_l$ its faces of dimension $d-1$. Let $S_i = \{x - y; x, y \in S, \tilde{y} \in \tilde{S} \cap F_i\} \subset \mathbb{Z}^n \oplus H$.

Let L be any lattice, corresponding to the semigroup $S \subset G(S)$, as in [5], we will define another semigroup $S' = \cap S_i \subset G(S)$ such that the semigroup ring $K[S']$ is the S_2-fication of the semigroup ring $K[S] := R/I_L$. Moreover if S is simplicial then $K[S']$ is the Macaulayfication of the semigroup ring $K[S] := R/I_L$. This extends to the lattice case a theorem of [5]. First we extends some preliminary results from [10], to the lattice case, the proofs are very similar and we let it to the reader.

1. Let $\bar{S} = \{z \in G(S), \exists p \in \mathbb{N}^*, pz \in S\}$, then $K[\bar{S}]$ is the normalization of $K[S]$.

2. A semigroup $S \subset \mathbb{Z}^d \oplus H$, is called standard if the following conditions are satisfied:

   (a) $\bar{S} = G(S) \cap \mathbb{N}^d \oplus H$,

   (b) $S_{(i)} \neq S_{(j)}$ for $i \neq j$, where $S_{(i)} = \{x \in S; x_i = 0\}$ and $x = (x_1, \ldots, x_l, h)$, with $h \in H$.

   (c) $\operatorname{rank}_{\mathbb{Z}} G(S_{(i)}) = \operatorname{rank}_{\mathbb{Z}} G(S) - 1,\ i = 1, \ldots, l$.

   By the Hochster's transformation, see [10], there is an standard semigroup $T(S)$ isomorphic to S, also by this transformation $T(S') = T(S)'$. So we can assume that our semigroup S is standard.

3. The polynomial ring R has two gradings, it is $G(S) = \mathbb{Z}^m/L = \mathbb{Z}^d \oplus H$-graded: two monomials $\underline{x}^\alpha, \underline{x}^\beta$ have the same grading if and only if the vector $\alpha - \beta \in L$, the lattice ideal I_L is $\mathbb{Z}^d \oplus H-$graded. The polynomial ring R is $\mathbb{Z}^d$-graded by grouping all homogeneous elements with the same $\mathbb{Z}^d$-graded component.

   Example 4 *The minimal primes ideals of I_L are $\mathbb{Z}^d$-graded , but not necessarily $\mathbb{Z}^d \oplus H-$ graded. Let $I = (x^2 - y^2) \subset K[x, y]$, in this case $L = \mathbb{Z}(2, -2)$ and the isomorphism $\mathbb{Z}^2/L \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, is given by $(a, b) \mapsto (a + b, b \bmod 2)$; it follows that deg$(x) = (1, 0)$, deg$(y) = (1, \bar{1})$. The minimal primary decomposition of I is given by $(x^2 - y^2) = (x - y)(x + y)$.*

   Let A be any arbitrary subset of $G(S)$, we will denote by $K[A]$ the $K-$vector space spanned by A in $K[G(S)]$. If $A + S \subset A$, we will call A an $S-$ideal. A proper subset P of S is a prime ideal if P is an $S-$ideal and $S \setminus P$ is additively closed. Every $G(S)-$graded prime ideal $\underline{p}$ of $K[S]$ is exactly of the form $K[P]$ for some prime ideal P of S and the homogeneous localization $K[S]_{\underline{p}}$ is isomorphic to $K[S - (S \setminus P)]$.

4. Let $S \subset \mathbb{Z}^d \oplus H$ be a standard semigroup with torsion, and let I be a nonempty subset of $[1, l]$, set $P_I = \{x \in S; x_i > 0$ for some $i \in I\}$ and $p_I = K[P_I]$. Then the set $\{P_I\}$ is the set of prime ideals of S (see the proof of the next lemma). Moreover $K[S]/p_{\{i\}} = K[S_{(i)}]$ and if $J \subset I$ then $P_J \subset P_I$. This implies that $p_{\{1\}}, \ldots, p_{\{l\}}$ are the unique $G(S)-$graded prime ideals of height one of $K[S]$.

The following Lemma shows that the extension from the toric case to the lattice case is non trivial:

Lemma 2 *Let $\mathcal{P} \subset K[S]$ be an $\mathbb{Z}^d$-graded prime ideal of height > 0. Then $\mathcal{P}$ is $G(S)-$graded and $\mathcal{P} = p_I$ for some non empty subset I.*

Proof .- First, let remark that if $z_1, z_2 \in K[S]$ are two pure monomials with the same $\mathbb{Z}^d$-grade, then $z_1^h - z_2^h = 0$, where h is the order of the group H, and this imply that for any prime ideal $\mathbf{p}$ in $K[S]$ there exists ξ a $h-$root of unity such that $z_1 + \xi z_2 \in \mathbf{p}$

We prove that $\mathcal{P}$ contains one monomial element $t^{g_1}\underline{u}^{g_2}$ for some $(g_1, g_2) \in S$.

Since $\mathrm{ht}(\mathcal{P}) \geq 1$, and because I_L is unmixed, $\mathcal{P}$ contains Q an associated prime of I_L and a non zero divisor z for $K[S]$. Now let $z \in \mathcal{P}$ be a non zero divisor, we can assume that z is $\mathbb{Z}^n-$homogeneous, if z is not monomial we can write it as a sum of monomials $z = \lambda_1 z_1 + \ldots + \lambda_r z_r$, with coefficients $\lambda_i \in K$, then for any $i = 1, \ldots, r$ there exists $h-$roots of unity ξ_i, such that $z_i + \xi_i z_1 \in Q \subset \mathcal{P}$, but

$$z = \sum_{i=1}^{r} \lambda_i (z_i + \xi_i z_1) - (\sum_{i=1}^{r} \lambda_i \xi_i) z_1,$$

and since $z \notin Q$ we have $\sum_{i=1}^{r} \lambda_i \xi_i \neq 0$, which implies that $z_1 \in \mathcal{P}$, and we are done.

The same proof shows that for any non empty set I, if $\mathcal{P} \not\subset p_I$, then we can choose a monomial element $z \in \mathcal{P} \setminus p_I$.

Let I be the set of integers $i \in [1, \ldots, l]$ such that $p_{\{i\}}$ is contained in $\mathcal{P}$, we will prove that I is non empty and $p_I = \mathcal{P}$.

It is clear that $p_I \subset \mathcal{P}$, remark that if $I = [1, \ldots, l]$, then $\mathcal{P}$ contains the unique graded maximal ideal of $K[S]$. So we can assume that I is a proper subset of $[1, \ldots, l]$. Suppose a contrario that there exist $z \in \mathcal{P} \setminus p_I$, (If I is empty choose z any monomial non zero divisor), we can assume that z is pure monomial, and if $z = \underline{t}^a \underline{u}^b$, then $a_i = 0$ for all $i \in I$, for any $j \notin I$ choose a monomial $\underline{t}^{c^j} \underline{u}^{d^j} \in p_{\{j\}} \setminus \mathcal{P}$, let $c = \sum_{j \notin I} c^j, d = \sum_{j \notin I} d^j$, then $c_j > 0$ for any $j \notin I$, and there exist a positive integer p such that $p(c, d) - (a, b) \in \mathbb{N}^l \oplus H \cap G = \bar{S}$, and for some positive integer k , $kp(c, d) - k(a, b) \in S$. It follows then that $\prod_{i \notin I} (\underline{t}^{c^j} \underline{u}^{d^j})^{pk} \in \mathcal{P}$ and $\underline{t}^{c^j} \underline{u}^{d^j} \in \mathcal{P}$, for some j. A contradiction.

Theorem 5 *Assume that the semigroup (eventually with torsion) S is standard. Let $G(S)$ be the group generated by S of rank d, let $S' := \cap_{i=1}^{l} (S - (S \setminus P_i))$ be a subsemigroup of $G(S)$, where $S \setminus P_i$ consist of the elements in S, which the i coordinate is 0. Then*

$$K[S'] = \cap_{i=1}^{l} K[S]_{(p_{\{i\}})},$$

and $K[S']$ satisfies the condition S_2. Let remark that $K[S]_{(p_{\{i\}})}$ is a homogeneous localization and the intersection is taken in the localization $T^{-1}K[S]$, where T is the set of all pure monomials, also since I_L is a lattice ideal any monomial is a non zero divisor for $K[S]$.

We also have an exact sequence :

$$0 \longrightarrow K[S] \longrightarrow K[S'] \longrightarrow K[S' \setminus S] \longrightarrow 0,$$

and $\dim K[S' \setminus S] \leq d - 2$. Moreover if S is simplicial then $K[S']$ is a Cohen-Macaulay ring.

Proof .- It follows from [5], p.244, that the property S_k holds for a $\mathbb{Z}^d$ graded module M if and only if

$$\text{depth } M_{(p)} \geq \min \{k, \dim M_{(p)}\}$$

for any $\mathbb{Z}^d$ homogeneous prime ideal p.

As a consequence the ring $\cap_{\mathrm{ht}(p)=1} K[S]_{(p)}$, where p runs over all $\mathbb{Z}^d$ homogeneous prime ideals in $K[S]$ of height one, satisfies the condition S_2. Now the above lemma proves that $\{p_{\{1\}}, \ldots, p_{\{l\}}\}$ are all the $\mathbb{Z}^d$ homogeneous prime ideals in $K[S]$ of height one and then

$$K[S'] = \cap_{i=1}^{l} K[S]_{(p_{\{i\}})},$$

satisfies the condition S_2. Also we have that $K[S]_{(p_{\{i\}})} = K[S']_{(p_{\{i\}})}$ and since the module $K[S' \setminus S]$ is $\mathbb{Z}^d$-graded we get $\dim K[S' \setminus S] \leq d-2$.

If S is simplicial, let $x_1, \ldots, x_d$ be the variables in R corresponding to the extreme rays of $\mathcal{C}_S$, then S' is also simplicial and $x_1, \ldots, x_d$ are parameters for both $K[S], K[S']$, since S' satisfies the condition S_2 we have that any pair x_i, x_j is a regular sequence in $K[S']$, if we have a relation $fx_i = \sum_{j<i} f_j x_j$ then because of the grading we certainly have $fx_i = f_j x_j$ for some j, this implies that the sequence $x_1, \ldots, x_d$ is a regular sequence in $K[S']$, so $K[S']$ is a Cohen-Macaulay ring.

As a Corollary we have

Corollary 2 *Let $L \subset \mathbb{N}^m$ be a positive lattice of rank r, set $\dim R/I_L = d = m-r$. If* $\operatorname{depth} R/I_L = d-1$ *then* $\dim K[S'-S] = d-2$, $\operatorname{depth}\ D^{d-1}(K[S]) = \operatorname{depth}\ K_{K[S]} - 2$, *and the following are equivalent:*

1. *$D^d(D^d(K[S]))$ is Cohen-Macaulay.*
2. *the canonical module of $K[S]$ is Cohen-Macaulay.*
3. *the module $D^{d-1}(K[S])$ is Cohen-Macaulay.*

The proof is immediate from Theorem 2.

Corollary 3 *Let $S \subset \mathbb{N}^n \oplus H$ be a simplicial finitely generated semigroup of rank d, then*

1. *$D^d(D^d(K[S]))$ is Cohen-Macaulay*
2. *the canonical module of $K[S]$ is Cohen-Macaulay*

We review the following example from [7], Example B.1:

Example 5 *Let K be a field, A the affine semigroup ring*

$$K[a, b, c, d, e^2, e^3, ade, bde, cde, d^2e].$$

We can see immediately that $K[S'] = K[a, b, c, d, e]$ and then it is a Macaulayfication of A, and we get the following exact sequence (see also [7]),

$$0 \longrightarrow A \longrightarrow K[a, b, c, d, e] \longrightarrow C[-1] \longrightarrow 0,$$

where $C = A/(ad, bd, cd, d^2, e^2, e^3, ade, bde, cde, d^2e)$ has dimension three. It follows that the Non Cohen-Macaulay locus of A is the support of C.

Example 6 *The following example is a toric ring of codimension two and dimension 4, which canonical module is not Cohen-Macaulay. The ideal $I_L \subset K[a, b, c, d, e, f] = R$ has the following generators:*

$$ab^4c - de^3f^2,\ bc^3d^3 - a^2e^2f^3,\ c^2d^4e - a^3b^3f,\ b^5c^4d^2 - ae^5f^5,$$
$$a^4b^7 - cd^5e^4f,\ c^5d^7 - a^5b^2ef^4,\ b^9c^5d - e^8f^7.$$

Let $0 \longrightarrow G \xrightarrow{\phi} F$ be the last term of a resolution of $A := S/I_L$, σ be the transpose of ϕ, then $F \xrightarrow{\sigma} G \longrightarrow D^3(A) \longrightarrow 0$ is a presentation of module $D^3(A)$, a quick computation by Macaulay gives that

$$\sigma = \begin{pmatrix} e^2f^2 & -bc & 0 & d & -a & 0 & 0 & 0 & 0 & 0 \\ ab^3 & -de & -f & 0 & 0 & c & 0 & 0 & 0 & 0 \\ c^2d^3 & -a^2f & 0 & 0 & 0 & 0 & e & -b & 0 & 0 \\ 0 & 0 & 0 & b^4c & -e^3f^2 & 0 & 0 & 0 & -d & a \end{pmatrix}$$

and that the module $D^3(A)$ has dimension 2, but depth $D^3(A) = 1$. *So the canonical module of S/I_L is not Cohen-Macaulay, in fact* depth $K_A = 3$.

In what follows we will write I instead I_L.

Theorem 6 *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring, let $A = R/I$ be a lattice ring, of codimension two and dimension d. If I is minimally generated by 4 generators, then $D^{d-1}(A)$ is a complete intersection. In particular, the canonical ring K_A is Cohen-Macaulay of dimension d, and the S_2-fication is a Macaulayfication of A. The Non-Cohen-Macaulay locus of A is the support of a Cohen-Macaulay module of dimension $d-2$.*

Proof .- The resolution of A, follows from [9] Construction 5.2:

$$0 \longrightarrow R \xrightarrow{\phi} R^4 \longrightarrow R^4 \longrightarrow R \longrightarrow A \longrightarrow 0$$

where σ the transpose of ϕ is given by:

$$\sigma = \begin{pmatrix} -x^{\mathbf{s}} & x^{\mathbf{t}} & x^{\mathbf{r}} & -x^{\mathbf{p}} \end{pmatrix}$$

where all monomials have disjoints supports. Then the entries of σ define a complete intersection, that is $D^{d-1}(A)$ is a complete intersection. The rest of the proof follows from Lemma 2.

Question Let $R = K[x_1, \ldots, x_{d+2}]$ be a polynomial ring, let $A = R/I$ be a lattice ring of codimension two and dimension d, is it true that $D^{d-1}(A)$ has non zero divisors?

5 Simplicial lattices ideals of height 2

Let K be a field and $R := K[y, z, x_1, \ldots, x_n]$ the ring of polynomials in the variables $y, z, x_1, \ldots, x_n$. Let a_i, b_i, c_i $1 \le i \le n$ be naturals numbers satisfying the conditions:

$$a_i \neq 0, (b_i, c_i) \neq 0 \forall i, (b_1, ..., b_n) \neq 0, (c_1, ..., c_n) \neq 0$$

For $i = 1, \ldots, n$ let $\mathbf{d_i} = a_i \mathbf{e_i}$, where $\mathbf{e_n}, \ldots, \mathbf{e_n}$ is the canonical basis of $\mathbb{N}^n$, and $\mathbf{a_1} = (b_1, ..., b_n), \mathbf{a_2} = (c_1, ..., c_n)$. Let H be a finite abelian group and $h_1, \ldots, h_{n+2} \in H$ that generates it. Let S be the subsemigroup of $\mathbb{N}^n \oplus H$ generated by

$$(\mathbf{d_1}, \mathbf{h_1}), \ldots, (\mathbf{d_n}, \mathbf{h_n}), (\mathbf{a_1}, \mathbf{h_{n+1}}), (\mathbf{a_2}, \mathbf{h_{n+2}}).$$

Definition 1 *A simplicial lattice ideal of height two is the lattice ideal $I_L \subset R$, where:*

$$L = \{\mathbf{w} \in \mathbb{Z}^{n+2}, w_1(\mathbf{d_1}, \mathbf{h_1}) + \ldots + w_n(\mathbf{d_n}, \mathbf{h_n}) + w_{n+1}(\mathbf{a_1}, \mathbf{h_{n+1}}) + w_{n+2}(\mathbf{a_2}, \mathbf{h_{n+2}}) = 0\}.$$

We remark that the last two coordinates of vectors in L, determine all the lattice L. More precisely, consider the group morphism:

$$\Phi : \mathbb{Z}^2 \longrightarrow \mathbb{Z}/a_1\mathbb{Z} \times \ldots \times \mathbb{Z}/a_n\mathbb{Z} \quad (s, p) \mapsto (sb_1 - pc_1, \ldots, sb_n - pc_n)$$

The lattice L is completely determined by the rank two sublattice :

$$\tilde{L} \subset Ker(\Phi) := \{(s, p) \in \mathbb{Z}^{\mathbf{2}} \ / \ sb_i - pc_i \equiv 0 \ mod \ a_i, \forall i = 1, \ldots, n\}.$$

$$\tilde{L} = \{(s, p) \in \mathbb{Z}^{\mathbf{2}} \ / \ s(\mathbf{a_1}, \mathbf{h_{n+1}}) - p(\mathbf{a_2}, \mathbf{h_{n+2}}) \in \ \mathbb{Z}(\mathbf{d_1}, \mathbf{h_1}) + \ldots + \mathbb{Z}(\mathbf{d_n}, \mathbf{h_n})\}.$$

Remark 2 *To any vector $(s,p) \in \tilde{L}$ with $s \geq 0$ we associate a unique binomial $B_{(s,p)} \in I_L$ in the following way: for any $i = 1, \ldots, n$, let v_i be the unique integer such that $sb_i - pc_i = v_i a_i$. We define the vectors $\mathbf{v}_+, \mathbf{v}_- \in \mathbb{N}^n$ by $\mathbf{v}_{+,i} = max\{v_i, 0\}, \mathbf{v}_{-,i} = max\{-v_i, 0\}$ and we must distinguish two cases:*

- *i) if $s \geq 0$ and $p \geq 0$ then $B_{(s,p)} = z^s \underline{x}^{\mathbf{v}_-} - y^p \underline{x}^{\mathbf{v}_+}$,*
- *ii) If $s \geq 0$ and $p < 0$ then $B_{(s,p)} = z^s y^{-p} \underline{x}^{\mathbf{v}_-} - \underline{x}^{\mathbf{v}_+}$.*

Let D_i be the line $D_i = \{(s,p) \in \mathbb{R}^2 \quad | \quad sb_i - pc_i = 0\}$. From now on, we suppose that the variables $x_1, ..., x_n$ are indexed in such a way that the slopes of the lines D_i are in increasing order.

Lemma 3 *Consider $B = M_1 - M_2 \in I_L$ a binomial, M_1, M_2 without common factors. We can write B in only one of the followings forms:*

1. *$z^s - y^p x_1^{v_1} \ldots x_n^{v_n}$, $s > 0$ $v_i \geq 0$ $\forall i$.*
2. *$y^p - z^s x_1^{v_1} \ldots x_n^{v_n}$, $p > 0$, $v_i \geq 0$ $\forall i$.*
3. *$y^p z^s - x_1^{v_1} \ldots x_n^{v_n}$, $p, s > 0$ $v_i > 0$ $\forall i$.*
4. *$z^s x_1^{v_1} \ldots x_k^{v_k} - y^p x_{k+1}^{v_{k+1}} \ldots x_n^{v_n}$, $v_i \geq 0$, $p, s > 0$, and $\exists 1 \leq i_1 \leq k$, $k+1 \leq i_2 \leq n$ / $v_{i_1}, v_{i_2} \neq 0$. In other words if $(\mathbf{v}, s, p) \in L$, with $s, p > 0$, and $\mathbf{v} = (v_1, \ldots, v_n)$ there exist k such that $v_i < 0$ for all $i < k$ and $v_i \geq 0$ for all $i \geq k$.*

As a consequence we have the following lemma:

Lemma 4 *1) There is no non trivial binomial in I_L, of the type: $z^s y^p \underline{x}^{\mathbf{v}_-} - \underline{x}^{\mathbf{v}_+}$ with $s \geq 0$, $p \geq 0$, and $\mathbf{v}_- \neq 0$.*
2) Consider an equality (where every fraction is reduced):

$$\frac{z^s P_1(\underline{x})}{\underline{x}^{\mathbf{v}}} = \frac{y^p P_2(\underline{x})}{\underline{x}^{\mathbf{w}}}.$$

- *If there exist an index i_1 such that $v_{i_1} > 0, w_{i_1} = 0$, then for all $j \geq i_1$, $v_j > w_j$.*
- *If there exist an index i_2 such that $v_{i_2} = 0, w_{i_2} > 0$, then for all $j \leq i_2$, $v_j < w_j$.*

The following proposition is an extension of [8], to the lattice case.

Proposition 1 *We can describe a fan decomposition of $\mathbf{R}^2_+$, more precisely we have vectors $\varepsilon_{-1}, \varepsilon_0, ..., \varepsilon_{m+1} \in \tilde{L} \cap \mathbf{Z}^2_+$ such that*

- *$\varepsilon_{-1} = (s_{-1}, 0), \varepsilon_0 = (s_0, p_0)$, with $0 \leq s_0 \leq s_{-1}$.*
- *Consider the Euclidean algorithm to compute the $gcd(s_{-1}, s_0)$:*

$$\begin{array}{rcl} s_{-1} & = & q_1 s_0 - s_1 \\ s_0 & = & q_2 s_1 - s_2 \\ & \cdots & \\ s_{m-1} & = & q_{m+1} s_m \\ s_{m+1} & = & 0 \end{array}$$

$$q_i \geq 2 \ , \ s_i \geq 0 \ \ \forall i$$

Let p_i be the sequence of integers defined by

$$p_{i+2} \quad = \quad q_{i+2} p_{i+1} - p_i \qquad -1 \leq i \leq m-1$$

then $\varepsilon_i = (s_i, p_i)$.

- *$\varepsilon_i, \varepsilon_{i+1}$ is a basis of $\tilde{L}$ and $det(\varepsilon_i, \varepsilon_{i+1}) = p_0 s_{-1} > 0$.*

Note that the existence of the basis $\varepsilon_{-1}, \varepsilon_0$ is provided by [2], page 62.

Definition 2 *Let $r_{j,i}$ be the sequence of integers defined by*

$$r_{j,i} \quad = \quad (s_i b_j - p_i c_j)/a_j \qquad -1 \leq i \leq m+1,\ 1 \leq j \leq n$$

and $\mathbf{r}_i$ the vector with coordinates $r_{j,i}$.

Lemma 5 *1) Any of the sequences $s_i, p_i, r_{j,i},\ 1 \leq j \leq n$ satisfy the recurrent relation:*

$$v_{i+2} = q_{i+2} v_{i+1} - v_i \text{ for } -1 \leq i \leq m-1.$$

2) The sequences $s_i, r_{j,i}$ (for all j) are strictly decreasing but the sequence p_i is strictly increasing.
3) Set ν (resp. μ) the greatest integer j such that $\mathbf{r}_j = \mathbf{r}_{j,+}$ (resp. the smallest integer j such that $\mathbf{r}_j = -\mathbf{r}_{j,-}$), then $-1 \leq \nu \leq \mu \leq m$.
4) $\text{supp } \mathbf{r}_{i+1,+} \subset \text{supp } \mathbf{r}_{i,+}$.

Theorem 7 *1) The ring R/I_L is arithmetically Cohen-Macaulay if and only if $\mu = \nu$. In this case the ideal I_L is generated by:*

$$\begin{array}{rcl} F & = & z^{s_\nu} \ - \ y^{p_\nu} \underline{x}^{\mathbf{r}_\nu} \\ G & = & y^{p_{\nu+1}} \ - \ z^{s_{\nu+1}} \underline{x}^{-\mathbf{r}_{\nu+1}} \\ H & = & z^{s_\nu - s_{\nu+1}} y^{p_{\nu+1} - p_\nu} \underline{x}^{\mathbf{r}_\nu \ - \mathbf{r}_{\nu+1}} \end{array}$$

2) If R/I_L is not arithmetically Cohen-Macaulay the ideal I_L is generated by $\tau := 3 + (q_{\nu+2} - 1) + ... + (q_{\mu+1} - 1)$ equations:

$$\begin{array}{rcl} z^{s_\nu} & - & y^{p_\nu} \underline{x}^{\mathbf{r}_\nu} \\ y^{p_{\nu+1} - p_\nu} z^{s_\nu - s_{\nu+1}} & - & \underline{x}^{\mathbf{r}_\nu - \mathbf{r}_{\nu+1}} \\ z^{s_{\nu+1}} \underline{x}^{\mathbf{r}_{\nu+1,-}} & - & y^{p_{\nu+1}} \underline{x}^{\mathbf{r}_{\nu+1,+}} \\ y^{2p_{\nu+1} - p_\nu} z^{s_\nu - 2s_{\nu+1}} & - & \underline{x}^{\mathbf{r}_\nu - 2\mathbf{r}_{\nu+1}} \\ & \cdots & \\ y^{(q_{\nu+2}-1)p_{\nu+1} - p_\nu} z^{s_\nu - (q_{\nu+2}-1)s_{\nu+1}} & - & \underline{x}^{\mathbf{r}_\nu - (q_{\nu+2}-1)\mathbf{r}_{\nu+1}} \\ z^{s_{\nu+2}} \underline{x}^{\mathbf{r}_{\nu+2,-}} & - & y^{p_{\nu+2}} \underline{x}^{\mathbf{r}_{\nu+2,+}} \\ & \cdots & \\ & \cdots & \\ z^{s_\mu} \underline{x}^{\mathbf{r}_{\mu,-}} & - & y^{p_\mu} \underline{x}^{\mathbf{r}_{\mu,+}} \\ y^{2p_\mu - p_{\mu-1}} z^{s_{\mu-1} - 2s_\mu} & - & \underline{x}^{\mathbf{r}_{\mu-1} - 2\mathbf{r}_\mu} \\ & \cdots & \\ y^{(q_{\mu+1}-1)p_\mu - p_{\mu-1}} z^{s_{\mu-1} - (q_{\mu+1}-1)s_\mu} & - & \underline{x}^{\mathbf{r}_{\mu-1} - (q_{\mu+1}-1)\mathbf{r}_\mu} \\ z^{s_{\mu+1}} \underline{x}^{\mathbf{r}_{\mu+1,-}} & - & y^{p_{\mu+1}} \end{array}$$

They form a Groebner's basis for the reverse lexicographic order with respect to $z < y < x_1 < \ldots < x_n$.

Proof .- Note that the proof given in [8], pp.1089, applies here without restriction. We outline the proof of 2): it consist to prove that the leading term of any binomial in I_L for the reverse lexicographic order with respect to $z < y < x_1 < \ldots < x_n$ is a factor of the leading term of some binomial in the above list. For example, let B be a binomial corresponding to the lattice point $(\mathbf{v}, s, p)$ with $p \geq 0, s \geq 0$. By the fan decomposition of $\mathbb{R}_+ \times \mathbb{R}_+$, there exists some $i \geq -1$ such that $(p, s) = \alpha\varepsilon_i + \beta\varepsilon_{i+1}$, with intehers $\alpha > 0, \beta \geq 0$, this imply $\mathbf{v} = \alpha\mathbf{r}_i + \beta\mathbf{r}_{i+1}$. We need to consider three cases:

- if $i < \nu$ then the coordinates of $\mathbf{r}_i, \mathbf{r}_{i+1}$ are all positive, so the leading term of B is z^s but $s = \alpha s_i + \beta s_{i+1} \geq s_\nu$.
- By a similar argument if $i \geq \mu + 1$ then the coordinates of $\mathbf{r}_i, \mathbf{r}_{i+1}$ are all negative, so the leading term of B is y^p but $p = \alpha p_i + \beta p_{i+1} \geq p_{\mu+1}$.
- if $\nu \leq i \leq \mu$ then the leading term of B is $z^s \underline{x}^{\mathbf{v}_-}$ which is a factor of $z^{s_i} \underline{x}^{\mathbf{r}_{i,-}}$.

If B is a binomial corresponding to the lattice point $(\mathbf{v}, s, p)$ with $p < 0, s \geq 0$. We argue with similar arguments using the fan decomposition of $\mathbb{R}_+ \times \mathbb{R}_-$ (that is every two consecutive vectors is a basis of $\tilde{L}$), given by the sequence of vectors

$$\varepsilon_{-1} - \varepsilon_0, ..., \varepsilon_{-1} - (q_1 - 1)\varepsilon_0 = \varepsilon_0 - \varepsilon_1, ..., \varepsilon_0 - (q_2 - 1)\varepsilon_1 = \varepsilon_1 - \varepsilon_2, ...,$$

$$\varepsilon_{m-1} - (q_m - 1)\varepsilon_m = \varepsilon_m - \varepsilon_{m+1}, -\varepsilon_{m+1}.$$

6 Macaulayfication of codimension two simplicial toric rings

The aim of this section consist to give an explicit description of the semigroup S' such that $K[S']$ is the Macaulayfication of the simplicial semigroup ring of codimension two $K[S]$. (see Theorem 5):

We recall that $S' = \cap_{i=1}^{l}(S - (S \setminus P_i))$ is a subsemigroup of $G(S)$, where $S \setminus P_i$ consist of the elements in S, which the i coordinate is 0. the ring

$$K[S'] = \cap_{i=1}^{l} K[S]_{(p_{\{i\}})},$$

is a Cohen-Macaulay ring, where $K[S]_{(p_{\{i\}})}$ is a homogeneous localization and the intersection is in the localization $T^{-1}K[S]$, where T is the set of all pure monomials. Remark that since I_L is a lattice ideal any monomial is a non zero divisor for $K[S]$. Any simplicial group is trivially standard. In what follows we will write I instead of I_L.

Remark 3 *Since S is simplicial of codimension two, every element in $S-(S\setminus P_i)$ can be viewed as a quotient of monomials $\dfrac{M(y,z,\underline{x})}{N(y,z,\underline{x})}$ where M, N are monomials with disjoints supports and $N \notin p_{\{i\}}$, we notice that*

$$p_{\{i\}} = \begin{cases} (x_i, y, z) & \textit{if } b_i \neq 0 \textit{ and } c_i \neq 0, \\ (x_i, y) & \textit{if } b_i = 0 \textit{ and } c_i \neq 0, \\ (x_i, z) & \textit{if } b_i \neq 0 \textit{ and } c_i = 0. \end{cases} \quad (3)$$

Lemma 6 *Let $E \in \cap_{i=1}^{n}(S - (S \setminus P_i))$, then for each i we can write $E = \dfrac{z^{\alpha_i} y^{\beta_i} P_i(\underline{x})}{Q_i(\underline{x})}$, such that x_i is not in the support of Q_i.*

Proof .- The assertion is clear if $b_i \neq 0$ and $c_i \neq 0$ for all $i = 1, ..., n$. If $b_i = 0$ and $c_i \neq 0$ then $s_j b_i - p_j c_i = -p_j c_i \leq 0$ and we have equality if and only if $i = -1$, this implies $\nu = -1, p_\nu = 0$. Regarding the order introduced in the variables $x_1, ..., x_n$ by the lemma 3, we can suppose that there exist natural integers k, l such that $b_1 = \ldots, b_k = 0, a_{n-l} = \ldots a_n = 0$ and that k, l are the biggest possible. It will be enough to prove the Lemma for $i \leq k$ and $0 < k < n$. Let $E \in \cap_{i=1}^{n}(S - (S \setminus P_i)), E = \dfrac{y^{\beta_i} P_i(\underline{x})}{Q_i(\underline{x}) z^{\alpha_i}}$ where x_i is not in the support of Q_i, and $\alpha_i > 0$. On the other hand for any $k < j < n - l$ we have $E = \dfrac{z^{\alpha_j} y^{\beta_j} P_j(\underline{x})}{Q_j(\underline{x})}$, where x_j is not in the support of Q_j remark that x_i belongs to the support of Q_j, otherwise we have finish our proof, we can also assume that P_j and Q_j have disjoint support, this gives us the following element in I:

$$y^{\beta_i} P_i(\underline{x}) Q_j(\underline{x}) - z^{\alpha_j + \alpha_i} y^{\beta_j} P_j(\underline{x}) Q_i(\underline{x})$$

Since x_i appears in the left side of this equality but no in the right side, we must have $\beta_j > \beta_i$. More precisely we write $Q_j(\underline{x}) = x_i^{\gamma_i} \tilde{Q}_j(\underline{x})$, $P_i(\underline{x}) = x_i^{\delta_i} \tilde{P}_i(\underline{x})$, and since the couple $(\alpha_i + \alpha_j, \beta_j - \beta_i)$ belongs to the lattice $\tilde{L}$, there exist integers A, B such that:

$$(\alpha_i + \alpha_j, \beta_j - \beta_i) = A(s_{-1}, 0) + B(s_0, p_0)$$

this implies that $\beta_j - \beta_i = Bp_0$. We have the following elements in I

$$z^{s_0} \underline{x}^{\mathbf{r}_{0,-}} - y^{p_0} \underline{x}^{\mathbf{r}_{0,+}}, z^{Bs_0} \underline{x}^{B\mathbf{r}_{0,-}} - y^{Bp_0} \underline{x}^{B\mathbf{r}_{0,+}}$$

this implies $\gamma_i + \delta_i = B\mathbf{r}_{0,-,i}$, and we have the following equality:

$$\frac{z^{Bs_0} \underline{x}^{B\hat{\mathbf{r}}_{0,-}} x^{\delta_i}}{\underline{x}^{B\mathbf{r}_{0,+}}} = \frac{y^{Bp_0}}{{x_i}^{\gamma_i}} = \frac{y^{\beta_j - \beta_i}}{{x_i}^{\gamma_i}}$$

where we have set $\mathbf{r}_{0,-,i}$ for the $i-$coordinate of the vector $\mathbf{r}_{0,-}$ and $\hat{\mathbf{r}}_{0,-}$is the vector $\mathbf{r}_{0,-}$ with the $i-$coordinate equal to zero. Finally we have

$$E = \frac{z^{\alpha_j} y^{\beta_i} P_i(\underline{x})}{\tilde{Q}_j(\underline{x})} \times \frac{z^{Bs_0} \underline{x}^{B\hat{\mathbf{r}}_{0,-}} x^{\delta_i}}{\underline{x}^{B\mathbf{r}_{0,+}}}$$

and x_i is not in the support of the denominator, and we are done.

Lemma 7 *Let $E \in \cap_{i=1}^{n}(S - (S \setminus P_i))$. For each i we write $E = \dfrac{z^{\alpha_i} y^{\beta_i} P_i(\underline{x})}{Q_i(\underline{x})}$, where x_i is not in the support of Q_i, and we can assume that P_i and Q_i have disjoint support. Then:*

1. *For all i, we can assume that $\alpha_i < s_\nu$ and $\beta_i < p_{\mu+1}$. The equality $\dfrac{y^{\beta_i} P_i(\underline{x})}{Q_i(\underline{x})} = \dfrac{y^{\beta_j} P_j(\underline{x})}{Q_j(\underline{x})}$, such that x_i is not in the support of Q_i and x_j is not in the support of Q_j, implies $i = j$ and this equality is an identity. The same is true for z.*

2. *If there exists some index i such that $\alpha_i = \beta_i = 0$ then $E \in S$.*

3. *We can write $E = z^\alpha y^\beta E'$ where $E' \in S'$, where α is the minimum of all the α_i and β is the minimum of all the β_i. In particular we can assume that there exist indexes i, j such that $\alpha_i = 0$ and $\beta_j = 0$.*

4. If $E = \dfrac{z^{\alpha} P(\underline{x})}{Q(\underline{x})}$, where P and Q have disjoint support, then x_1 is not in the support of Q.

Proof .-

1. Suppose that $\dfrac{y^{\beta_i} P_i(\underline{x})}{Q_i(\underline{x})} = \dfrac{y^{\beta_j} P_j(\underline{x})}{Q_j(\underline{x})}$ such that x_i is not in the support of Q_i, x_j is not in the support of Q_j, $i \neq j$ and $\beta_j \geq \beta_i$. It follows that $\tilde{P}_i(\underline{x})\tilde{Q}_j(\underline{x}) - y^{\beta_j - \beta_i}\tilde{P}_j(\underline{x})\tilde{Q}_i(\underline{x})$ belongs to I where $\tilde{P}_l = P_l / \gcd(P_i, P_j)$, $\tilde{Q}_l = Q_l / \gcd(Q_i, Q_j)$. Since $0 \leq \beta_j - \beta_i < p_{\mu+1}$ such element cannot exists in I, and we are done.

2. Suppose that $E = \dfrac{P_i(\underline{x})}{Q_i(\underline{x})}$, such that x_i is not in the support of Q_i but $Q_i \neq 1$ and P_i, Q_i have disjoint support. Let x_j be in the support of Q_i, then we can write $E = \dfrac{z^{\alpha_j} y^{\beta_j} P_j(\underline{x})}{Q_j(\underline{x})}$, such that x_j is not in the support of Q_j. It follows that $z^{\alpha_j} y^{\beta_j} P_j(\underline{x}) Q_i(\underline{x}) - P_i(\underline{x}) Q_j(\underline{x})$ belongs to I. We get a contradiction since x_j is not in the support of $P_i Q_j$.

3. It is clear that $\dfrac{z^{\alpha_i - \alpha} y^{\beta_i - \beta} P_i(\underline{x})}{Q_i(\underline{x})} = \dfrac{z^{\alpha_j - \alpha} y^{\beta_j - \beta} P_j(\underline{x})}{Q_j(\underline{x})}$ in the field of fractions of $K[S]$. We set $E' = \dfrac{z^{\alpha_i - \alpha} y^{\beta_i - \beta} P_i(\underline{x})}{Q_i(\underline{x})}$, now it is clear that $E' \in S'$ and $E = z^{\alpha} y^{\beta} E'$.

4. Suppose that x_1 is in in the support of Q, then we can write $\dfrac{z^{\alpha} P(\underline{x})}{Q(\underline{x})} = \dfrac{z^{\alpha_1} y^{\beta_1} P_1(\underline{x})}{Q_1(\underline{x})}$, such that x_1 is not in the support of Q_1 and $\beta_1 > 0$. It follows then that $z^{\alpha} P(\underline{x}) Q_1(\underline{x}) - z^{\alpha_1} y^{\beta_1} P_1(\underline{x}) Q(\underline{x}) \in I$, but lemma 4, implies that $\alpha > \alpha_1$, and we get that $z^{\alpha - \alpha_1} P(\underline{x}) Q_1(\underline{x}) - y^{\beta_1} P_1(\underline{x}) Q(\underline{x}) \in I$, and x_1 is the support of $P_1(\underline{x}) Q(\underline{x})$ but not in the support of $P(\underline{x}) Q_1(\underline{x})$, applying again lemma 4, we get a contradiction since $\alpha - \alpha_1 < s_{\nu}$.

Theorem 8 *1. Any element in the minimal basis in I of the type*

$$z^{s_{\nu+l}} \underline{x}^{\mathbf{r}_{\nu+l,-}} - y^{p_{\nu+l}} \underline{x}^{\mathbf{r}_{\nu+l,+}},$$

for $1 \leq l \leq \mu - \nu$, gives rise to a non trivial element

$$E_l = \frac{y^{p_{\nu+l}}}{\underline{x}^{\mathbf{r}_{\nu+l,-}}} = \frac{z^{s_{\nu+l}}}{\underline{x}^{\mathbf{r}_{\nu+l,+}}} \in S'.$$

2. Any element $E \in S'$ which can be written as

$$\frac{y^{\beta}}{\underline{x}^{\mathbf{v}_-}} = \frac{z^{\alpha}}{\underline{x}^{\mathbf{v}_+}},$$

where $\mathbf{v}_+, \mathbf{v}_-$ have disjoint support, belongs to the semigroup generated by S and the elements E_l, for $1 \leq l \leq \mu - \nu$.

3. Any element $E \in S'$ which can be written as

$$\frac{y^{\beta} P(\underline{x}))}{\underline{x}^{\mathbf{v}_-}} = \frac{z^{\alpha} Q(\underline{x})}{\underline{x}^{\mathbf{v}_+}},$$

where $\mathbf{v}_+, \mathbf{v}_-$ have disjoint support, belongs to the semigroup generated by S and the elements E_l for $1 \leq l \leq \mu - \nu$.

Proof .-

1. It is clear that $E_l = \dfrac{y^{p_{\nu+l}}}{\underline{x}^{\mathbf{r}_{\nu+l,-}}} \dfrac{z^{s_{\nu+l}}}{\underline{x}^{\mathbf{r}_{\nu+l,+}}} \in S'$. We have $E_l \notin S$ since $p_{\nu+l} < p_{\mu+1}$.

2. Let $E \in S'$ such that $\dfrac{y^\beta}{\underline{x}^{\mathbf{v}_-}} = \dfrac{z^\alpha}{\underline{x}^{\mathbf{v}_+}}$, where $\mathbf{v}_+, \mathbf{v}_-$ have disjoint support. It follows that $y^\beta \underline{x}^{\mathbf{v}_+} - z^\alpha \underline{x}^{\mathbf{v}_-}$ belongs to I then $(\alpha, \beta) \in \ker \Phi$ and there exist positive integers k, λ_1, λ_2 such that
$$(\alpha, \beta) = \lambda_1(s_k, p_k) + \lambda_2(s_{k+1}, p_{k+1})$$
and as consequence of this
$$v_j = \lambda_1 r_{j,k} + \lambda_2 r_{j,k+1} \text{ for all } 1 \leq j \leq n.$$
We recall that if $r_{j,l} > 0$ for some j, l then $r_{m,l} > 0$ for all $m > j$ and that supp $\mathbf{r}_{k+1,+} \subset$ supp $\mathbf{r}_{k,+}$. By lemma 3, there exist δ such that $v_l < 0$ for all $l < \delta$ and $v_l \geq 0$ for all $l \geq \delta$. Let supp $\mathbf{r}_{k,+} = \{l_1, \ldots, n\}$, supp $\mathbf{r}_{k+1,+} = \{l_2, \ldots, n\}$, with $l_1 \leq l_2$. It follows that $l_1 \leq \delta \leq l_2$ then we can write
$$E = \frac{z^\alpha}{\underline{x}^{\mathbf{v}_+}} = \Big(\frac{x_{l_1}^{r_{l_1,k}} \ldots x_{\delta-1}^{r_{\delta-1,k}} z^{s_k}}{\underline{x}^{r_{k,+}}}\Big)^{\lambda_1} \Big(\frac{x_\delta^{-r_{\delta,k+1}} \ldots x_{l_2-1}^{-r_{l_2-1,k+1}} z^{s_{k+1}}}{\underline{x}^{r_{k+1,+}}}\Big)^{\lambda_2}$$
so
$$E = (x_{l_1}^{r_{l_1,k}} \ldots x_{\delta-1}^{r_{\delta-1,k}} x_\delta^{-r_{\delta,k+1}} \ldots x_{l_2-1}^{-r_{l_2-1,k+1}})^{\lambda_2} E_k^{\lambda_1} E_{k+1}^{\lambda_2}.$$

3. If $E = \dfrac{y^\beta P(\underline{x})}{\underline{x}^{\mathbf{v}_-}} = \dfrac{z^\alpha Q(\underline{x})}{\underline{x}^{\mathbf{v}_+}}$, where $\mathbf{v}_+, \mathbf{v}_-$ have disjoint support, then after division by the common factor of P, Q we can assume that they have disjoint support. But then $P(\underline{x})\underline{x}^{\mathbf{v}_+}$ and $Q(\underline{x})\underline{x}^{\mathbf{v}_-}$ have disjoint support. It follows that the element $E' := \dfrac{y^\beta}{Q(\underline{x})\underline{x}^{\mathbf{v}_-}} = \dfrac{z^\alpha}{P(\underline{x})\underline{x}^{\mathbf{v}_+}}$ belongs to S' and we can write $E = P(\underline{x})Q(\underline{x})E'$, the assertion follows from the previous item.

Theorem 9 *Any element $E \in S'$ belongs to the semigroup generated by S and the elements E_l for $1 \leq l \leq \mu - \nu$.*

Proof .- Let $E \in S'$ be a non trivial element, by lemma 7, item 4 we can write $E = \dfrac{z^\alpha P_1(\underline{x})}{x_1^{a_1^1} \ldots x_n^{a_n^1}}$ with $a_1^1 = 0$. Let i_1 be the biggest integer such that $a_j^1 = 0$ for $j < i_1$ but $a_{i_1}^1 > 0$. Since $E \in S' = \cap_{i=1}^n (S - (S \setminus P_i))$ we can write $E = \frac{y^{\beta_1} z^{\alpha_1} P_3(\underline{x})}{x_1^{a_1^3} \ldots x_n^{a_n^3}}$ with $a_{i_1}^3 = 0$ and $\beta_1 > 0$. It then follows that
$$z^\alpha P_1(\underline{x}) x_1^{a_1^3} \ldots x_n^{a_n^3} - y^{\beta_1} z^{\alpha_1} P_3(\underline{x}) x_1^{a_1^1} \ldots x_n^{a_n^1} \in I,$$
and lemma 4 implies that $0 < \alpha_1 < \alpha$, so we have that
$$z^{\alpha-\alpha_1} P_1(\underline{x}) x_1^{a_1^3} \ldots x_n^{a_n^3} - y^{\beta_1} P_3(\underline{x}) x_1^{a_1^1} \ldots x_n^{a_n^1} \in I.$$
Since $a_{i_1}^1 > 0, a_{i_1}^3 = 0$ then for all $j \geq i_1$, $a_j^1 \geq a_j^3$ by lemma 4.

Thus we can write the equality:

$$\frac{z^{\alpha-\alpha_1}P_1(\underline{x})}{x_1^{a_1^1}\dots x_{i_1}^{a_{i_1}^1}x_{i_1+1}^{a_{i_1+1}^1-a_{i_1+1}^3}\dots x_n^{a_n^1-a_n^3}} = \frac{y^{\beta_1}P_3(\underline{x})}{x_1^{a_1^3}\dots x_{k_1-1}^{a_{k_1-1}^3}} \tag{4}$$

Since the denominators have disjoint support, this equality gives one element $F_1 \in S'$ that belongs to the semigroup generated by S and $E_1,\dots,E_{\mu-\nu}$. Then we have that:

$$E = F_1\frac{z^{\alpha_1}}{x_{i_1+1}^{a_{i_1+1}^3}\dots x_n^{a_n^3}} \tag{5}$$

Now either $a_j^3 = 0$ for all $j > i_1$, and in this case we have finished the proof of the theorem, or there exist $i_2 > i_1$ such that $a_j^3 = 0$ for all $i_1 \leq j < i_2$, but $a_{i_2}^3 > 0$. Since $E \in S' = \cap_{i=1}^n(S-(S\setminus P_i))$ we can write $E = \dfrac{y^{\beta_2}z^{\alpha_2}P_4(\underline{x})}{x_1^{a_1^4}\dots x_n^{a_n^4}}$ with $a_{i_2}^4 = 0$ and $\beta_2 \geq 0$. We have the following element in I_L

$$y^{\beta_2}z^{\alpha_2}P_4(\underline{x})\underline{x}^{(\mathbf{a}_3-\mathbf{a}_4)_+} - y^{\beta_1}z^{\alpha_1}P_3(\underline{x})\underline{x}^{(\mathbf{a}_3-\mathbf{a}_4)_-}$$

First since $\alpha_1 < s_\nu, \alpha_2 < s_\nu, \beta_1 < p_{\mu+1}, \beta_2 < p_{\mu+1}$, we must have $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$.

Now suppose that $\beta_2 < \beta_1$, we have two cases:

1. If $\alpha_1 > \alpha_2$ then Lemma 4, implies that $a_i^3 > a_i^4$ for all i but $a_{i_1}^3 = 0$, this is a contradiction.
2. If $\alpha_1 < \alpha_2$ since $a_{i_2}^3 > 0, a_{i_2}^4 = 0$ by Lemma 4, we get $a_i^3 > a_i^4$ for all $i \leq i_2$ but $a_{i_1}^3 = 0$, this is a contradiction.

So we have $\beta_2 > \beta_1$, if we assume that $\alpha_1 < \alpha_2$ since $a_{i_2}^3 > 0, a_{i_2}^4 = 0$ by lemma 4 we have $a_i^3 > a_i^4$ for all $i \leq i_2$ but $a_{i_1}^3 = 0$, this is a contradiction. Finally we get $\beta_2 > \beta_1$ and $\alpha_1 > \alpha_2$.

Using lemma 4, we argue as before and we get that for all $j \geq i_2$, $a_j^3 \geq a_j^4$. We have the following equality:

$$\frac{z^{\alpha_1-\alpha_2}}{x_{i_2}^{a_{i_2}^3-a_{i_2}^4}\dots x_n^{a_n^3-a_n^4}} = \frac{y^{\beta_2-\beta_1}P_4(\underline{x})}{P_3(\underline{x})x_1^{a_1^4}\dots x_{i_2-1}^{a_{i_2-1}^4}}.$$

This equality defines one element $F_2 \in S'$ that belongs to the semigroup generated by S and $E_1,\dots,E_{\mu-\nu}$, and we have

$$E = F_1F_2\frac{z^{\alpha_2}}{x_{i_2+1}^{a_{i_2+1}^4}\dots x_n^{a_n^4}}. \tag{6}$$

We can continue and we can write

$$E = F_1F_2\dots F_m,$$

where $F_1, F_2, \dots, F_m$ belong to the semigroup generated by S and $E_1,\dots,E_{\mu-\nu}$. This ends the proof of the theorem.

Example 7 *Let k be a non zero natural number, and consider the simplicial toric variety defined parametrically by:*

$$x_1 = u_1^{2k},\dots,x_k = u_k^{2k}, y = u_1^{k+1}u_2u_3\dots u_k, z = u_1u_2^{k+1}u_3\dots u_k$$

It is a codimension two variety in $\mathbb{P}^{k+1}$. Let I_k be the vanishing ideal of this variety. We apply the algorithm described in proposition 1 to find a system of generators of I_k:

$$
\begin{array}{rcl}
y^{2k} & - & x_1^{k+1}x_2x_3\dots x_k \\
z^2x_1 & - & y^2x_2 \\
y^{2k-2}z^2 & - & x_1^kx_2^2x_3\dots x_k \\
y^{2k-4}z^4 & - & x_1^{k-1}x_2^3x_3\dots x_k \\
 & \dots & \\
y^2z^{2k-2} & - & x_1^2x_2^kx_3\dots x_k \\
z^{2k} & - & x_1x_2^{k+1}x_3\dots x_k
\end{array}
$$

In order to get the Macaulayfication we must consider the element:

$$\frac{y^2}{x_1} = \frac{z^2}{x_2} = u_1^2u_2^2u_3^2\dots u_k^2.$$

The Macaulayfication will be the semigroup ring :

$$K[S'] = K[u_1^{2k}, \dots, u_k^{2k}, u_1^{k+1}u_2u_3\dots u_k, u_1u_2^{k+1}u_3\dots u_k, u_1^2u_2^2u_3^2\dots u_k^2].$$

In fact it is easy to check that

$$K[S'] = K[x_1, x_2, x_3, \dots, x_k, y, z, w]/(z^2 - x_2w, y^2 - x_1w, w^k - x_1x_2x_3\dots x_k),$$

and it is a complete intersection.

Example 8 *We can apply our methods to some non toric cases. The (non-toric) variety* $V \subset \mathbb{A}^7$ *defined by*

$$x_1 = s^4 + t^4;\ x_2 = s^2tu;\ x_3 = s^3t;\ x_4 = st^3;\ x_5 = su^3;\ x_6 = s^2t^2v;\ x_7 = v$$

is a generalized f-variety, not locally Cohen-Macaulay, and $\dim V = 4$.

Let V_1 *be the variety defined by*

$$x_1 = s^4 + t^4;\ x_2 = s^2tu;\ x_3 = s^3t;\ x_4 = st^3;\ x_5 = su^3;\ x_6 = s^2t^2;\ x_7 = v.$$

Let $K[V]$ *and* $K[V_1]$ *be respectively the coordinate rings of* V *and* V_1. *It is immediate to check that* V_1 *is a complete intersection, and therefore it is arithmetically Cohen-Macaulay, in fact* V_1 *is the Macaulayfication of* V.

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